HOMEWORK 5

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1. Suppose f is non-constant analytic in $\mathbb{D}$ and $|f(z)| \leq M$ on $\mathbb{D}$. Prove that the number of zeros of f in the disc of radius $1/4$, centered at 0, does not exceed

$$\frac{1}{\log 4} \log \left| \frac{M}{f(0)} \right|.$$

Hint: Use the Blaschke products factorization.

2. Let f be analytic in $\mathbb{D}$ and satisfy $|f(z)| \to 1$ as $|z| \to 1$. Prove f is rational.

Hint: Use the Blaschke products factorization and the Maximum Principle.

3. For $z, w \in \mathbb{D}$ recall the pseudo-hyperbolic distance $\rho(z, w) = |z - w|/|1 - \bar{w}z|$. Verify the so-called World's Greatest Identity:

$$1 - \rho(z, w)^2 = \frac{(1 - |z|^2)(1 - |w|^2)}{|1 - \bar{w}z|^2}.$$

4. (a) Show that the pseudo-hyperbolic neighborhood of radius $r < 1$ of the segment $[0, 1)$ is contained in a Stolz angle with vertex at 1.

Hint: Write the generic point z in this neighborhood as $z = (\zeta + x)/(1 + x\zeta)$ for some $|\zeta| < r$ and some $x \in [0, 1)$.

(b) A horodisk is a disk internally tangent to $\partial\mathbb{D}$ at 1. Show that for $t > 0$, the set

$$H(t) = \left\{ z \in \mathbb{D} : \frac{1 - |z|^2}{|1 - z|^2} > t \right\}$$

is a horodisk and find a formula $r = r(x)$ for the radius of a hyperbolic disk $\Delta(x, r)$ so that as $x \uparrow 1$ the disk $\Delta(x, r)$ approximates $H(t)$.

Hint: For the last part use the World's Greatest Identity.

5. Let f be analytic on a convex domain $D \subset \mathbb{C}$. Show that if $\operatorname{Re} f'(z) > 0$ for every $z \in D$, then f is injective on D.

Hint: Computing difference quotients using the Fundamental Theorem of Calculus.

6. Let f be analytic on all of $\mathbb{C}$ and suppose $f(\sqrt[n]{n}) \in \mathbb{R}$ for $n = 1, 2, 3, \ldots$. Show that $f(\mathbb{R}) \subset \mathbb{R}$.

Hint: Study $g(z) := \overline{f(\bar{z})}$.

7. Define the complex partial differential operators

$$\partial f = \frac{\partial f}{\partial z} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right) \qquad \text{and} \qquad \bar{\partial} f = \frac{\partial f}{\partial \bar{z}} := \frac{1}{2}\left(\frac{\partial f}{\partial x} + \frac{1}{i}\frac{\partial f}{\partial y}\right)$$

Compute $\partial |z|$ and $\bar{\partial}\partial(1-|z|^2)/|1-z|^2$.

8. Consider the line segment $[-i,i]$, the circular arc on $\partial\mathbb{D}$ that lies in $\{z \in \mathbb{C} : \operatorname{Re} z \geq 0\}$, and the circular arc on $\partial\mathbb{D}$ that lies in $\{z \in \mathbb{C} : \operatorname{Re} z \leq 0\}$. Parametrize them with constant speed paths γ_j, $j = 1,2,3$. Then compute

$$\int_{\gamma_j} |z| dz$$

for $j = 1,2,3$.

9*. Given a metric space (X,d), a continuous map $\gamma : [a,b] \longrightarrow (X,d)$ is called a path in X. Its total variation is called the length:

$$\operatorname{length}(\gamma) := \sup_{a=t_0 \leq t_1 \cdots \leq t_N = b} \sum_{k=0}^{N-1} d(\gamma(t_k), \gamma(t_{k+1})),$$

where the supremum is taken over all possible partitions with N arbitrary. The path γ is called rectifiable if $\operatorname{length}(\gamma) < \infty$. The length function of γ is the function $s_\gamma : [a,b] \longrightarrow [0, \operatorname{length}(\gamma)]$ defined by $s_\gamma(t) := \operatorname{length}(\gamma\mid_{[a,t]})$.

(a) Show that s_γ is increasing (not necessarily 'strictly'), and check that for any $a \leq t < s \leq b$:

$$d(\gamma(t), \gamma(s)) \leq \operatorname{length}(\gamma\mid_{[t,s]}) = s_\gamma(s) - s_\gamma(t).$$

(b) Prove that if γ is a rectifiable path in X, then s_γ is continuous.

Hint: use the continuity of γ to show that if $s_\gamma(s) - s_\gamma(t) > \delta$, then there are $t = a_0 < a_1 < \cdots < a_k < s$ such that $\sum_{j=0}^{k-1} d(\gamma(a_{j+1}), \gamma(a_j)) > \delta$.

(c) Prove that a path γ in X is absolutely continous if and only if its length function s_γ is absolutely continuous.

10*. (Continuation of previous problem). Consider the one-sided inverse of s_γ defined by $s_\gamma^{-1}(u) := \sup\{s : s_\gamma(s) = u\}$ and define the arc-length parametrization of γ to be the path $\gamma_s(t) : [0, \operatorname{length}(\gamma)] \to X$ given by $\gamma_s(u) := \gamma(s_\gamma^{-1}(u))$.

Show that γ_s is Lipschitz continuous and hence absolutely continuous.

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